Digital Signal Processing using MATLAB

4. The z – Transform

- Two shortcomings to the Fourier transform approach
  - 1. useful signals in practice for which the discrete-time Fourier transform does not exist [ex) *u(n), nu(n)*]
  - 2. the transient response of a system due to initial conditions or due to changing inputs cannot be computed using the discrete-time Fourier transform aproach
- z-Transform
  - Extension of the discrete-time Fourier transform

THE BILATERAL z-TRANSFORM

- 1. the z-transform of a sequence *x(n)*

$$X(z) \cong Z[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

  - ROC (region of convergence)
    - Set of z values for which *X(z)* exists

$$R_{x-} < |z| < R_{x+}$$

THE BILATERAL z-TRANSFORM

- 2. inverse z-Transform of a complex function *X(z)*

$$x(n) \cong Z^{-1}[X(z)] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

  - [c : counterclockwise contour encircling the origin and lying in the ROC]

THE BILATERAL z-TRANSFORM

- *Comments:*
  - 1. z (complex variable) : complex frequency

    $$z = |z| e^{jw}$$

    - $|z|$: attenuation
    - w : real frequency
  - 2. shape of the ROC is an open ring
    - Figure 4.1
    - May be $R_{x-} = 0,\ R_{x+} = \infty$

THE BILATERAL z-TRANSFORM

- 3. if) $R_{x+} < R_{x-}$
- then) ROC is a null space and z-transform does not exit
- 4. if) $|z|=1$ (or $z=e^{jw}$)
- then) unit circle
  - If) ROC contains the unit circle
  - Then) evaluate $X(z)$ on the unit circle

    $$X(z)|_{z=e^{jw}} = X(e^{jw}) = \sum_{n=-\infty}^{\infty} x(n)e^{-jw} = F[x(n)]$$

  - ∴DTFT is a special case of the z-Transform $X(z)$

EX

- Example 4.1 (positive-time sequence)
  - Let) $x_1(n) = a^n u(n), \quad 0 < |a| < \infty$
  - Then) $X_1(z) = \sum_0^{\infty} (\frac{a}{z})^n = \frac{1}{1 - az^{-1}}; \; if \; |\frac{a}{z}| < 1$

$$= \frac{z}{z-a}, \quad |z| > |a| \Rightarrow ROC_1 : |a| < |z| < \infty$$

$$R_{x-} \qquad R_{x+}$$

  - Note : $X(z)$ is rational function

$$X_1(z) \cong \frac{B(z)}{A(z)} = \frac{z}{z-a}$$

  - $B(z) = z$: numerator polynomial
  - $A(z) = z-a$: denominator polynomial
  - Root of $B(z)$: zeros of $X(z)$ ($X_1(z)$, $z=0$)
  - Root of $A(z)$: poles of $X(z)$ ($X_1(z)$, $z=a$)
- Pole-zero diagram in z-plane
  - FIGURE 4.2 (the ROC in ex4.1)
    - Zeros : 0
    - Poles : x

- Example 4.2 (negative-time sequence)
  - Let) $x_2(n) = -b^n u(-n-1), \quad 0 < |b| < \infty$
  - Then) $X_2(z) = -\sum_{-\infty}^{-1} b^n z^{-n} = -\sum_{-\infty}^{-1} (\frac{b}{z})^n = -\sum_{1}^{\infty} (\frac{z}{b})^n$

$$= 1 - \sum_{0}^{\infty} (\tfrac{z}{b})^n = 1 - \frac{1}{1 - z/b} = \frac{z}{z-b}, \quad ROC_2 : 0 < |z| < |b|$$

$$R_{x-} \qquad R_{x+}$$

  - FIGURE 4.3 (the ROC in ex4.2)

- Example4.3 (two-sided sequence)
  - Let) $x_3(n) = x_1(n) + x_2(n) = a^n u(n) - b^n u(-n-1)$
  - Then) $X_3(z) = \sum_{n=0}^{\infty} a^n z^{-n} - \sum_{-\infty}^{-1} b^n z^{-n}$

$$= \{\frac{z}{z-a}, \; ROC_1 : |z| > |a|\} + \{\frac{z}{z-b}, \; ROC_2 : 0 < |z| < |b|\}$$

$$= \frac{z}{z-a} + \frac{z}{z-b} \quad ROC_3 : ROC_1 \cap ROC_2$$

  - FIGURE 4.4 (the ROC in ex4.3)

PROPERTIES OF THE ROC

- 1. ROC is always bounded by a circle
- 2. $x_1(n)=a^n u(n)$ ($x_1(n)=0$, $n<n_0$)
  - Special case of a right-sided sequence
  - ROC (right-sided sequence) 범위는
    - always outside of a circle of radius R_{x-}
    - If) $n_0 \geq 0$
    - Then) right-sided sequence is causal sequence

- $3. x_2(n)=-b_n u(-n-1)$ ($x_2(n)=0$, $n>n_0$)
  - Special case of a left sided sequence
  - ROC (left-sided sequence) 범위는
    - Always inside of a circle of radius R_{x+}
    - If) $n_0 \leq 0$
    - Then) left-sided sequence is anticausal sequence
- 4. $x_3=x_1(n)+x_2(n)$
  - Two-sided sequence
  - ROC : always an open ring $R_{x-} < |z| < R_{x+}$

- 5. $n<n_1$, $n>n_2$ 에서 0 인 경우
  - : Finite-duration sequence
  - ROC : the entire z-plane
  - If) $n_1<0$
  - Then) $z=\infty$ is not in the ROC
  - If) $n_2>0$
  - Then) $z=0$ is not in the ROC
- 6. ROC cannot include a pole
- 7. there is at lest one pole in the boundary of a ROC of a rational $X(z)$
- 8. ROC is one contiguous region ; that is, the ROC does not come in pieces

IMPORTANT PROPERTIES OF THE z-TRANSFORM

- 1. linearity :
  - $z[a_1x_1(n)+a_2x_2(n)] = a_1X_1(z) + a_2X_2(z)$
  - ROC : $ROC_{x1} \cap ROC_{x2}$
- 2. sample shifting :
  - $z[x(n-n_0)] = z^{-n_0}X(z)$; ROC : ROC_x
- 3. frequency shifting :
  - $z[a^n x(n)]=X(z/a)$; ROC : ROC_x scaled by $|a|$

- 4. Folding :
  - $z[x(-n)] = X(1/z)$; ROC : Inverted ROCx
- 5. Complex conjugation :
  - $z[x^*(n)] = x^*(z^*)$; ROC : ROCx
- 6. Differentiation in the z- domain :
  - $Z[nx(n)] = -z\dfrac{dX(z)}{dz}$; ROC : ROCx

- 7. Multiplication :

$$Z[x_1(n)x_2(n)] = \frac{1}{2\pi j}\oint_c X_1(v)X_2(z/v)v^{-1}dv;$$

  - ROC : ROCx ∩ inverted ROCx2
  - c : closed contour that encloses the origin and lies in the common ROC
- 8. convolution :
  - $z[x_1(n)*x_2(n)] = X_1(z)X_2(z)$;
  - ROC : $ROC_{x1} \cap ROC_{x2}$

Example 4.4

- Let) $X_1(z) = 2 + 3z^{-1} + 4z^{-2}$

  $X_2(z) = 3 + 4z^{-1} + 5z^{-2} + 6z^{-3}$

  $X_3(z) = X_1(z)X_2(z)$?

- sol)

```
>> x1 = [2,3,4] ; x2 = [3,4,5,6];
>> x3 = conv(x1,x2)
```

```
x3 =    6    17    34    43    38    24
```

$$X_3(z) = 6 + 17z^{-1} + 34z^{-2} + 43z^{-3} + 38z^{-4} + 24z^{-5}$$

Example 4.5

- Let) $X_1(z) = x + 2 + 3z^{-1}$

  $X_2(z) = 2z^2 + 4z + 3 + 5z^{-1}$

  $X_3(z) = X_1(z)X_2(z)$?

- Sol)

```
>> x1 = [1,2,3] ; n1=[-1:1] ;
>> x2 = [3,4,5,6]; n2=[-2:1] ;
>> [x3,n3] = conv_m(x1, n1, x2, n2)
```

```
x3 =   3   10   22   28   27   18
n3 =  -3   -2   -1    0    1    2
```

$$X_3(z) = 2z^3 + 8z^2 + 17z + 23 + 19z^{-1} + 15z^{-2}$$

Deconvolution

- MATLAB

```
[p,r]=deconv(b,a)
```

- Compute the result of dividing *b* by *a* in a polynomial part *p* and a remainder *r*

SOME COMMON z-TRANSFORM PAIRS

$\delta(n)$	1	$\forall z$
$u(n)$	$\frac{1}{1-z^{-1}}$	$\|z\|>1$
$-u(n-1)$	$\frac{1}{1-z^{-1}}$	$\|z\|<1$
$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\|z\|<\|a\|$
$b^n u(n)$	$\frac{1}{1-bz^{-1}}$	$\|z\|<\|b\|$
$[a^n \sin w_0 n]u(n)$	$\frac{(a\sin w_0)z^{-1}}{1-(2a\cos w_0)z^{-1}+a^2z^{-2}}$	$\|z\|<\|a\|$
$[a^n \cos w_0 n]u(n)$	$\frac{1-(a\cos w_0)z^{-1}}{1-(2a\cos w_0)z^{-1}+a^2z^{-2}}$	$\|z\|>\|a\|$
$na^n u(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\|<\|a\|$
$-nb^n u(-n-1)$	$\frac{bz^{-1}}{(1-bz^{-1})^2}$	$\|z\|<\|b\|$

Example 4.6

$$x(n) = (n-2)(0.5)^{(n-2)} \cos[\tfrac{\pi}{3}(n-2)]u(n-2)$$

- Sol) Applying the ***sample-shift*** property

$$X(z) = Z[x(n)] = z^{-2} Z[n(0.5)^n \cos(\tfrac{\pi n}{3})u(n)]$$

- Applying the ***multiplication by a ramp*** property

$$X(z) = z^{-2}\{-z \frac{dZ[(0.5)^n \cos(\frac{\pi n}{3})u(n)]}{dz}\}$$

$$Applying\ Table\ 4.1 \quad Z[(0.5)^n \cos(\tfrac{\pi n}{3})u(n)] = \frac{1-(0.5\cos\frac{\pi}{3})z^{-1}}{1-2(0.5\cos\frac{\pi}{3})z^{-1}+0.25z^{-2}} \quad |z|>0.5$$

Hence

$$X(z) = -z^{-1}\frac{d}{dz}\{\frac{1-(0.5\cos\frac{\pi}{3})z^{-1}}{1-2(0.5\cos\frac{\pi}{3})z^{-1}+0.25z^{-2}}\} = -z^{-1}\{\frac{-0.25z^{-2}+0.5z^{-3}-0.0625z^{-4}}{1-z^{-1}+0.75z^{-2}-0.25z^{-3}+0.0625z^{-4}}\}$$

$$= \frac{-0.25z^{-3}+0.5z^{-4}-0.0625z^{-5}}{1-z^{-1}+0.75z^{-2}-0.25z^{-3}+0.0625z^{-4}}$$


```
?b=[0,0,0,0.25,-0.5,0.0625];
?a=[1,-1,0.75,-0.25,0.0625];
?[delta,n]=impseq(0,0,7)

delta =    1    0    0    0    0    0    0    0
n =        0    1    2    3    4    5    6    7

?x=filter(b,a,delta)
x = 0      0      0    0.2500   -0.2500   -0.3750   -0.1250    0.0781

?x=[(n-2).*(1/2).^(n-2).*cos(pi*(n-2)/3)].*stepseq(2,0,7)
x = 0      0      0    0.2500   -0.2500   -0.3750   -0.1250    0.0781
```

INVERSION OF THE z-TRANSFORM

- *Central Idea :* When *X(z)* is a rational function of z^{-1}, it can be expressed as a sum of simple (first-order) factors using the partial fraction expansion
- *Method :* Given

$$X(z) = \frac{b_0 + b_1 z^{-1} + \dots\dots + b_M z^{-M}}{1 + a_1 z^{-1} + \dots\dots + a_N z^{-N}} \quad , R_{x-} < |z| < R_{x+}$$

- Express it as

$$X(z) = \underbrace{\frac{b_0 + b_1 z^{-1} + \dots\dots + b_{N-1} z^{-(N-1)}}{1 + a_1 z^{-1} + \dots\dots + a_N z^{-N}}}_{\textit{proper rational part}} + \underbrace{\sum_{k=0}^{M-N} C_k z^{-k}}_{\textit{polynomial part if } M \ge N}$$

- Perform a partial traction expansion on the proper rational part of *X(z)* to obtain.

$$X(z) = \sum_{k=1}^{N} \frac{R_k}{1 - p_k z^{-1}} + \sum_{k=0}^{M-N} C_k z^{-k}$$

$$M \ge N$$

$$R_k = \frac{\tilde{b}_0 + \tilde{b}_1 z^{-1} + \dots\dots + \tilde{b}_{N-1} z^{-(N-1)}}{1 + a_1 z^{-1} + \dots\dots + a_N z^{-N}} (1 - p_k z^{-1}) \big|_{z = p_k}$$

p_k : kth pole of $X(z)$

R_k : residues at p_k

- If a pole p_k has multiplicity r
- then)

$$\sum_{l=1}^{r} \frac{R_{k,l} z^{-(l-1)}}{(1 - p_k z^{-1})^l} = \frac{R_{k,1}}{1 - p_k z^{-1}} + \frac{R_{k,2} z^{-1}}{(1 - p_k z^{-1})^2} + \dots\dots + \frac{R_{k,r} z^{-(r-1)}}{(1 - p_k z^{-1})^r}$$

- Write *x(n)* as

$$x(n)=\sum_{k=1}^{N} R_k Z^{-1}[\frac{1}{1-p_k z^{-1}}]+\sum_{k=0}^{M-N} C_k \delta(n-k)$$

$$M \ge N$$

- Finally, use the relation from Table 4.1 to complete

$$Z^{-1}[\frac{z}{z-p_k}]=\begin{cases} p_k^n u(n) & |z_k| \le R_{x-} \\ -p_k^n u(-n-1) & |z_k| \le R_{x+} \end{cases}$$

Example 4.7

- Inverse z-transform of $X(z)=\frac{z}{3z^2-4z+1}$
- Sol) Write

$$X(z)=\frac{z}{3(z^2-\frac{4}{3}z+\frac{1}{3})}\frac{\times z^{-2}}{\times z^{-2}}=\frac{\frac{1}{3}z^{-1}}{1-\frac{4}{3}z^{-1}+\frac{1}{3}z^{-2}}=\frac{\frac{1}{3}z^{-1}}{(1-z^{-1})(1-\frac{1}{3}z^{-1})}=\frac{\frac{1}{2}}{1-z^{-1}}-\frac{\frac{1}{2}}{1-\frac{1}{3}z^{-1}}$$

$$X(z)=\frac{1}{2}\left(\frac{1}{1-z^{-1}}\right)-\frac{1}{2}\left(\frac{1}{1-\frac{1}{3}z^{-1}}\right) \qquad X(z) \rightarrow z_1=1\ ,\quad z_2=\frac{1}{3}$$

- Figure 4.5

- a. ROC$_1$: $1<|z|<\infty$ (both pole are on the interior side of the ROC$_1$ $|z_1|\geq R_{x+}=1$ and $|z_2|\leq 1$)

$$x_1(n)=\frac{1}{2}u(n)-\frac{1}{2}\left(\frac{1}{3}\right)^n u(n) \quad \text{right-sided sequence}$$

- b. ROC$_2$: $0<|z|<1/3$ (both pole are on the exterior side of the ROC$_2$ $|z_1|\geq R_{x+}=1/3$ and $|z_2|\leq 1/3$)

$$x_2(n)=\frac{1}{2}\{-u(-n-1)\}-\frac{1}{2}\{-\left(\frac{1}{3}\right)^n u(-n-1)\}=\frac{1}{2}\left(\frac{1}{3}\right)^n u(-n-1)-\frac{1}{2}u(-n-1) \quad \text{left-sided sequence}$$

$$x_3(n)=-\frac{1}{2}u(-n-1)-\frac{1}{2}\left(\frac{1}{3}\right)^n u(n) \quad \text{two-sided sequence}$$

- c. ROC$_3$: $1/3<|z|<1$ (pole z_1 is on the exterior side of the ROC$_3$ $|z_2|\leq 1/3$)

MATLAB IMPLEMENTATION

- Residues : (MATLAB function)
  - Compute the residue part and the direct terms of a rational function in z^{-1}

$$X(z)=\frac{b_0+b_1z^{-1}+\dots\dots+b_Mz^{-M}}{1+a_1z^{-1}+\dots\dots+a_Nz^{-N}}=\frac{B(z)}{A(z)}$$

$$=\sum_{k=1}^{N}\frac{R_k}{1-p_kz^{-1}}+\sum_{k=0}^{M-N}C_kz^{-k}$$

$$M\geq N$$

[R, p, C]=residuez(b, a)

- Similarly

```
[b, a]=residuez(R, p, C)
```

- If) *p(k) = ……= p(k+r−1)* is pole multiplicity *r*,
- Then) $\frac{R_k}{1-p_k z^{-1}} + \frac{R_{k+1}}{(1-p_k z^{-1})^2} + \ldots.. + \frac{R_{k-r-1}}{(1-p_k z^{-1})^r}$

- Example 4.8 $X(z) = \frac{z}{3z^2 - 4z + 1}$
  - Sol)

$$X(z) = \frac{z}{(3z^2 - 4z + 1)} \frac{\times z^{-2}}{\times z^{-2}} = \frac{z^{-1}}{3 - 4z^{-1} + z^{-2}} = \frac{0 + z^{-1}}{3 - 4z^{-1} + z^{-2}}$$

  - Using MATLAB

```
b = [0,1];
a = [3,-4,1];
[R,p,C] = residuez(b,a)
```

```
R =     0.5000
       -0.5000
p =     1.0000
        0.3333
C =    []
```

$$X(z) = \frac{\frac{1}{2}}{1 - z^{-1}} - \frac{\frac{1}{2}}{1 - \frac{1}{3} z^{-1}}$$

```
[b,a] = residuez(R,p,C)
```

```
b =   -0.0000    0.3333
a =    1.0000   -1.3333    0.3333
```

$$X(z) = \frac{0 + \frac{1}{3} z^{-1}}{1 - \frac{4}{3} z^{-1} + \frac{1}{3} z^{-2}} = \frac{z^{-1}}{3 - 4z^{-1} + z^{-2}} = \frac{z}{3z^2 - 4z + 1}$$

Example 4.9 compute inverse z-transform

$$X(z) = \frac{1}{(1-0.9z^{-1})^2(1+0.9z^{-1})}, \quad |z| > 0.9$$

Sol)

```
b = 1; a = poly([0.9,0.9,-0.9])
a =    1.0000   -0.9000   -0.8100    0.7290
[R,p,c] = residuez(b,a)
R =   0.2500 + 0.0000i
      0.5000 - 0.0000i
      0.2500
p =   0.9000 + 0.0000i
      0.9000 - 0.0000i
     -0.9000
c =   []
```

$$X(z) = \frac{0.25}{1-0.9z^{-1}} + \frac{0.5}{(1-0.9z^{-1})^2} + \frac{0.25}{1+0.9z^{-1}}, \quad |z| > 0.9$$

$$= \frac{0.25}{1-0.9z^{-1}} + \frac{0.5}{0.9} z \frac{0.9z^{-1}}{(1-0.9z^{-1})^2} + \frac{0.25}{1+0.9z^{-1}}, \quad |z| > 0.9$$

Using TABLE 4.1

$$x(n) = 0.25(0.9)^n u(n) + \frac{5}{9}(n+1)(0.9)^{n+1} u(n+1) + 0.25(-0.9)^n u(n)$$

$$x(n) = 0.75(0.9)^n u(n) + 0.5n(0.9)^n u(n) + 0.25(-0.9)^n u(n)$$

<비교> b, a 계수를 이용해 filter 를 구한 것과, x(n)과 MATLAB을 이용해 비교한다.

```
[delta,n] = impseq(0,0,7) ;
x=filter(b,a,delta)
x =    1.0000    0.9000    1.6200    1.4580    1.9683
       1.7715    2.1258    1.9132
x=(0.75)*(0.9).^n+(0.5)*n.*(0.9).^n+ (0.25)*(-0.9).^n
x =    1.0000    0.9000    1.6200    1.4580    1.9683
       1.7715    2.1258    1.9132
```

Example 4.10 inverse z-transform

$$X(z)=\frac{1+0.4\sqrt{2}z^{-1}}{1-0.8\sqrt{2}z^{-1}+0.64z^{-2}}$$

Sol)

```
b = [1,0.4*sqrt(2)];
a=[1,-0.8*sqrt(2),0.64];
[R,p,C] = residuez(b,a)
% pole magnitudes
Mp=abs(p')
% pole angles in pi units
Ap=angle(p')/pi
```

```
R =   0.5000 - 1.0000i
      0.5000 + 1.0000i
p =  0.5657 + 0.5657i
      0.5657 - 0.5657i
C =    []

Mp =    0.8000    0.8000
Ap =   -0.2500    0.2500
```

$$X(z)=\frac{0.5+j}{1-|0.8|e^{-j\frac{\pi}{4}n}z^{-1}}-\frac{0.5-j}{1-|0.8|e^{j\frac{\pi}{4}n}z^{-1}}\ ,\ |z|>0.8$$

- Using TABLE 4.1

$$x(n)=(0.5+j)|0.8|^n\ e^{-j\frac{\pi}{4}n}u(n)+(0.5-j)|0.8|^n\ e^{j\frac{\pi}{4}n}u(n)$$

$$=|0.8|^n\ [0.5\{e^{-j\frac{\pi}{4}n}+e^{j\frac{\pi}{4}n}\}+j\{e^{-j\frac{\pi}{4}n}+e^{j\frac{\pi}{4}n}\}]u(n)$$

$$=|0.8|^n\ [\cos(\frac{\pi n}{4})+2\sin(\frac{\pi n}{4})]u(n)$$

- <비교> b, a 계수를 이용해 filter 를 구한 것과, x(n)과 MATLAB을 이용해 비교한다.

```
[delta,n] = impseq(0,0,7) ;
x=filter(b,a,delta)
x =    1.0000    1.6971    1.2800    0.3620   -0.4096
     -0.6951   -0.5243  -0.1483
x=((0.8).^n).*(cos(pi*n/4)+2*sin(pi*n/4))
x =    1.0000    1.6971    1.2800    0.3620   -0.4096
     -0.6951   -0.5243  -0.1483
```

SYSTEM REPRESENTATION IN THE z-DOMAIN

- H(z) : system function, z-domain function

- DEFINITION1 *The System Function*

$$H(z) \cong Z[h(n)] = \sum_{\infty}^{\infty} h(n) z^{-n}; \quad R_{h-} < |z| < R_{h+}$$

- Convolution property
- $Y(z) = H(z)X(z)$: ROC_y $ROC_h \cap ROC_x$

X(z) → H(z) → Y(z) = H(z)X(z)

- SYSTEM FUNTION FROM THE DIFFRENCE EQUATION REPRESENTATION
  - LTI system (described by a difference equation)

$$y(n) + \sum_{k=1}^{N} a_k y(n-k) = \sum_{l=0}^{M} b_l x(n-l)$$

- z-Transform

$$Y(z)+\sum_{k=1}^{N}a_k z^{-k}Y(z)=\sum_{l=0}^{M}b_l z^{-l}X(z)$$

$$H(z)\cong\frac{Y(z)}{X(z)}=\frac{\sum_{l=0}^{M}b_l z^{-l}}{1+\sum_{k=1}^{N}a_k z^{-k}}=\frac{B(z)}{A(z)}=\frac{b_0 z^{-M}(z^M+\ldots\ldots+\frac{b_M}{b_0})}{z^{-N}(z^{-N}+\ldots\ldots+a_N)}$$

$$H(z)=b_0 z^{N-M}\frac{\prod_{l=1}^{M}(z-z_l)}{\prod_{k=1}^{N}(z-p_k)}$$

z_l : system zeros
p_k : system poles

- MATLAB function
  - roots : determine zeros and poles of rational *H(z)*
  - poly : determaine polynomial coerrieients from its roots
  - zplane(b,a) :plots poles and zeros, given the numerator row vector b and the denominator row vector a.
    - 0 : zeros
    - X : poles

TRANSFER FUNCTION REPRESENTATION

- If) ROC of *H(z)* includes a unit circle ($z=e^{jw}$)
- Then) *H(z)* on the unit circle
- That is -> $H(e^{jw})$
- (frequency response, transfer function)
- From 4.21

$$H(z) = b_0 e^{j(N-M)w} \frac{\prod_1^M (e^{jw} - z_l)}{\prod_1^N (e^{jw} - p_k)}$$

- FIGURE 4.6 pole and zero vector

- Magnitude response function

$$|H(e^{jw})| = |b_0| \frac{|e^{jw} - z_1| \ldots |e^{jw} - z_M|}{|e^{jw} - p_1| \ldots |e^{jw} - p_N|}$$

- Phase response function

$$\angle H(e^{jw}) = [0\, or\, \pi] + [(N-M)w] + \sum_1^M \angle(e^{jw} - z_k) - \sum_1^M \angle(e^{jw} - p_k)$$

constant linear nonlinear

MATLAB IMPLEMENTATION

- freqz :
- [H, w] = freqz(b, a, N)
  - b ,a : numerator, denominator coefficients
  - N : N point equally spaced
  - w : N point frequency vector
  - H : N-point complex frequency response vector
- [H, w] = freqz(b, a, N, 'whole')
  - N point around the whole unit circle for compute
- H = freqz(b, a, w)
  - The frequency response at frequencies designated in vector w, normally between 0 and π

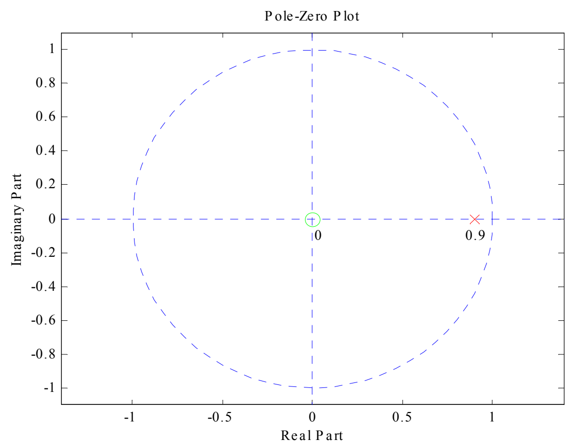
Example 4.11 given a causal system

$$y(n) = 0.9y(n-1) + x(n)$$

- A. find *H(z)* , pole-zero plot
  - Sol) from Table 4.1

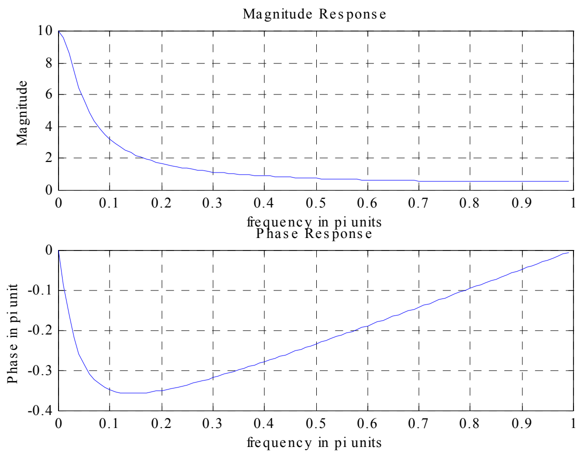
$$H(z) = \frac{1}{1-0.9z^{-1}} \; ; \; |z| > 0.9$$

```
b = [1,0]; a = [1, -0.9];
[H1,H2,H3] = zplane(b,a);
set(H1,'markersize',10,'color',[0,1,0]);
set(H2,'markersize',10,'color',[1,0,0]);
title('Pole-Zero Plot');
text(0.85,-0.1,'0.9');text(0.01,-0.1,'0');
```



- B. plot $|H(e^{jw})|$ and $\angle H(e^{jw})$

```
[H,w] =freqz(b,a,100);
magH = abs(H) ;
phaH = angle(H);
subplot(2,1,1) ; plot(w/pi, magH) ; grid
xlabel('frequency in pi units') ; ylabel('Magnitude') ;
title('Magnitude Response');
subplot(2,1,2) ; plot(w/pi, phaH/pi) ; grid
xlabel('frequency in pi units') ; ylabel('Phase in pi unit ') ;
title('Phase Response');
```



- C. determine the impulse response *h(n)*
  - Sol) from the z –transform Table 4.1

$$h(n) = Z^{-1}\left[\frac{1}{1-0.9z^{-1}}, |z| > 0.9\right] = (0.9)^n u(n)$$

Example 4.12 $H(z)=\dfrac{z+1}{z^2-0.9z+0.81}$

- A. transfer function representation
  - Sol) substituting $z=e^{jw}$ in $H(z)$

$$H(e^{jw})=\frac{e^{jw}+1}{e^{j2w}-0.9e^{jw}+0.81}=\frac{e^{jw}+1}{(e^{jw}-0.9e^{j\pi/3})(e^{jw}-0.9e^{-j\pi/3})}$$

- B. difference equation representation
  - Sol) H(z) = Y(z)/X(z)

$$\frac{Y(z)}{X(z)}=\frac{z+1}{z^2-0.9z+0.81}\left(\frac{z^{-2}}{z^{-2}}\right)=\frac{z^{-1}+z^{-2}}{1-0.9z^{-1}+0.81z^{-2}}$$

$$Y(z)-0.9z^{-1}Y(z)+0.81z^{-2}Y(z)=z^{-1}X(z)+z^{-2}X(z)$$

Inverse z transform $y(n)-0.9y(n-1)+0.81y(n-2)=x(n-1)+x(n-2)$

or $y(n)=0.9y(n-1)-0.81y(n-2)+x(n-1)+x(n-2)$

- C. impulse response representation.
  - Sol) using MATLAB

```
b=[0, 1, 1]; a=[1, -0.9,0.81];
[R,p,C] =residuez(b,a)

R =  -0.6173 - 0.9979i
     -0.6173 + 0.9979i
p =   0.4500 + 0.7794i
      0.4500 - 0.7794i
C =    1.2346
```

```
Mp=abs(p')
Mp =    0.9000    0.9000

Ap = angle(p')/pi
Ap =   -0.3333    0.3333
```

$$H(z) = 1.2346 + \frac{-0.6173 + j0.9979}{1 - |0.9|\, e^{-j\pi/3} z^{-1}} + \frac{-0.6173 - j0.9979}{1 - |0.9|\, e^{j\pi/3} z^{-1}}$$

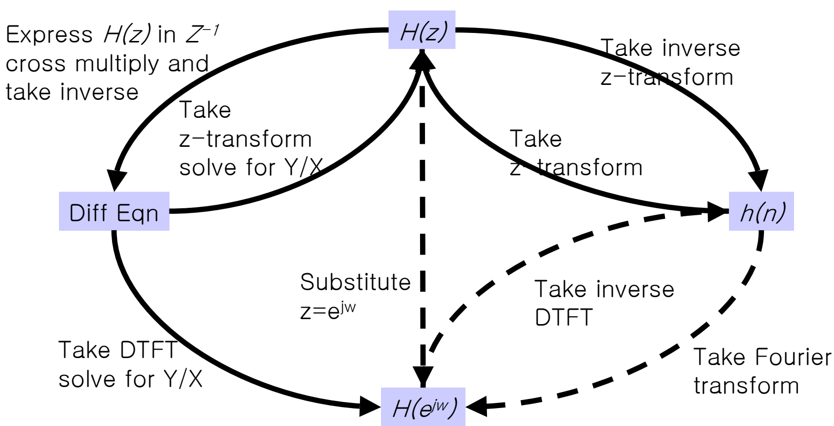
Hence from Table 4.1

$$h(n) = 1.2346\delta(n) + [(-0.6173 + j0.9979)\,|0.9|^n\, e^{-j\pi n/3} + (-0.6173 - j0.9979)\,|0.9|^n\, e^{j\pi n/3}]u(n)$$

$$= 1.2346\delta(n) + |0.9|^n\, [-1.2346\cos(\pi n/3) + 1.9958\sin(\pi n/3)]u(n)$$

$$= |0.9|^n\, [-1.2346\cos(\pi n/3) + 1.9958\sin(\pi n/3)]u(n-1)$$

RELATIONSHIPS BETWEEN SYSTEM REPRESENTATIONS



STABLITY AND CAUSALITY

- LTI Systems the BIBO stability
  - $\sum_{-\infty}^{\infty} |h(k)| < \infty$.
  - $H(e^{jw})$ exists
  - Unit circle $|z|=1$ must be in the ROC of *H(z)*
- THEOREM 2 *z-Domain LTI Stability*
  - An LTI system is stable if and only if the unit circle is in the ROC of *H(z)*

- THEOREM 3 *z-Domain causal LTI stability*
  - A causal LTI system is stable if and only if the system function *H(z)* has all its poles inside the unit circle.

Example 4.13 causal LTI system (by difference equation)

$$y(n) = 0.81y(n-2) + x(n) - x(n-2)$$

- A. the system function *H(z)*
  - Sol) from(4.20)

$$Y(z) - 0.81z^{-2}Y(z) = X(z) - z^{-2}X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-z^{-2}}{1-0.81z^{-2}} = \frac{1-z^{-2}}{(1+0.9z^{-1})(1-0.9z^{-1})}, |z| > 0.9$$

- B. the unit impulse response *h(n)*
  - Sol) using MATLAB

```
b=[1, 0, -1]; a=[1, 0, -0.81];
[R,p,C]=residuez(b,a)
R =   -0.1173
      -0.1173
p =    0.9000
      -0.9000
C =    1.2346
```

$$H(z) = 1.2346 - 0.1173\frac{1}{1+0.9z^{-1}} - 0.1173\frac{1}{1-0.9z^{-1}}, \; |z| > 0.9$$

from table 4.1

$$h(n) = 1.2346\delta(n) - 0.1173\{1+(-1)^n\}(0.9)^n u(n)$$

- C. unit step response v(n)
  - Sol) from table 4.1 $Z[u(n)] = \frac{1}{1-z^{-1}}, |z|>1$

$$V(z) = H(z)U(z) = \left[\frac{(1+z^{-1})(1-z^{-1})}{(1+0.9z^{-1})(1-0.9z^{-1})}\right]\left[\frac{1}{1-z^{-1}}\right], |z|>0.9 \cap |z|>1$$

$$= \frac{1+z^{-1}}{(1+0.9z^{-1})(1-0.9z^{-1})}, |z|>0.9$$

or

$$V(z) = 1.0556\frac{1}{1-0.9z^{-1}} - 1.0556\frac{1}{1+0.9z^{-1}}, \quad |z|>0.9$$

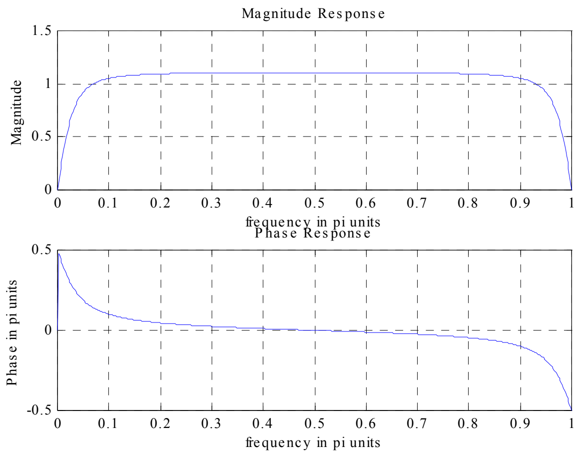
finally

$$v(n) = [1.0556(0.9)^n - 0.0556(-0.9)^n]u(n)$$

- D. the frequency response function $H(e^{jw})$, and plot its magnitude and phase over $0 \leq w \leq \pi$
  - Sol) substituting $z = e^{jw}$ in $H(z)$

$$H(e^{jw}) = \frac{1-e^{-j2w}}{1-0.81e^{-j2w}}$$

```
w=[0:1:500]*pi/500;
H=freqz(b,a,w);
magH=abs(H); phaH=angle(H);
subplot(2,1,1); plot(w/pi,magH); grid
xlabel('frequency in pi units'); ylabel('Magnitude')
title('Magnitude Response')
subplot(2,1,2); plot(w/pi,phaH/pi); grid
xlabel('frequency in pi units'); ylabel('Phase in pi units')
title('Phase Response')
```


Magnitude Response
Magnitude
1.5
1
0.5
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
frequency in pi units
Phase Response
Phase in pi units
0.5
0
-0.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
frequency in pi units